Midterm: • Wednesday Feb 27, in-class
(1:00 PM here)

• Will be on the same material as before (up through induction)

Equivalence Classes:

Ex: $A = \{$all living organisms$\}$.

Taxonomic Rank: Domain, Kingdoms, Phylum, ..., Genus, Species

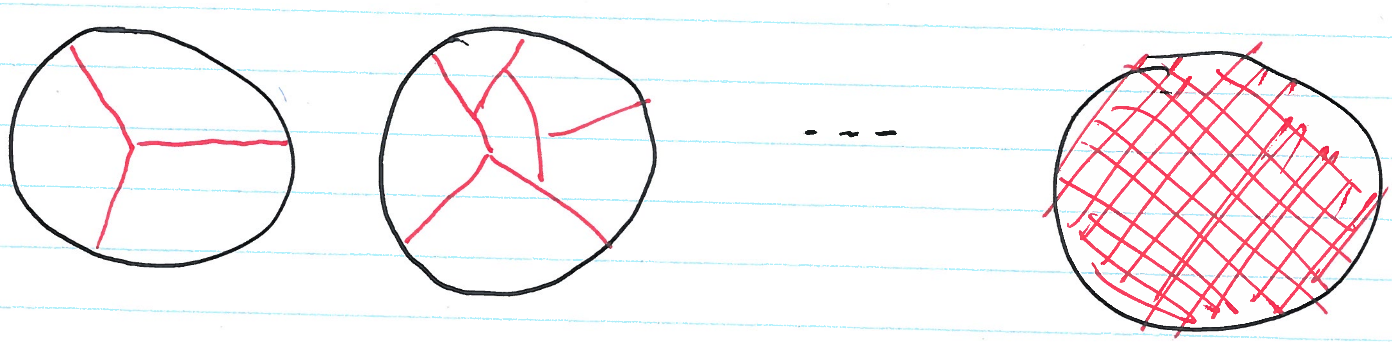
$\longleftarrow$ more general

$\longrightarrow$ more specific

Each taxonomic rank corresponds to an equivalence relation on A. (Ex. aRb if a, b have same genus)

Each one defines equivalence classes on A.

A



Def: Suppose A is a set and $R \subseteq A \times A$ is an equivalence relation on A. Then for each element $a \in A$, its <u>equivalence class</u> is the set

$$[a]_R = \{x \in A : xRa\}$$

(Sometimes we just write $[a]$ instead of $[a]_R$.)

Ex: Suppose $A = \mathbb{Z}$. Consider three relations

- aR_2b if $a \equiv b \pmod 2$
- aR_4b $\quad a \equiv b \pmod 4$
- aR_8b $\quad a \equiv b \pmod 8$.

R_2: Each integer $a \in \mathbb{Z}$ defines an eq. class $[a]$

$[0] = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

$[1] = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$

$[2] = \{\ldots, -4, -2, 0, 2, 4, \ldots\} = [0]$

In general, $[a] = \begin{cases} \{\text{even}\} & \text{if } a \text{ is even} \\ \{\text{odd}\} & \text{if } a \text{ is odd.} \end{cases}$

Fact: For any two elements $a, b \in A$,

$$[a] = [b] \iff aRb$$

For R_2 on $\mathbb{Z}$, there are two equivalence classes

Exercise: List the equivalence classes for R_4 and R_8.

For R_4: There are 4 equivalence classes.

$$\{\ldots, -8, -4, 0, 4, 8, \ldots\} = [0]_{R_4} = [4]_{R_4}$$

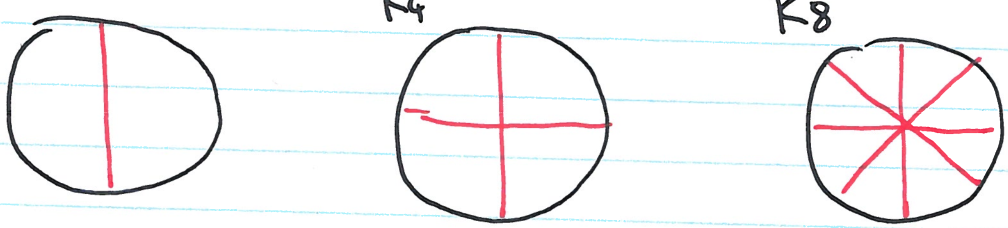
$$\{\ldots, -7, -3, 1, 5, 9, \ldots\} = [1]_{R_4}$$

$$\{\ldots, -6, -2, 2, 6, 10, \ldots\} = [2]_{R_4}$$

$$\{\ldots, -5, -1, 3, 7, 11, \ldots\} = [3]_{R_4}$$

pick some representatives

R_2 $\quad$ R_4 $\quad$ R_8



In general: Integers mod n have n equivalence classes.

Prop: $[a] = [b] \iff aRb$.

Pf: $\Rightarrow$: If $[a] = [b]$, then

$$\{x : xRa\} = \{x : xRb\}$$

So because aRa, it follows aRb.

$\Leftarrow$: Suppose aRb.

Pick any $x \in [a]$. Then xRa.

By transitivity, xRa and $aRb \Rightarrow xRb$.

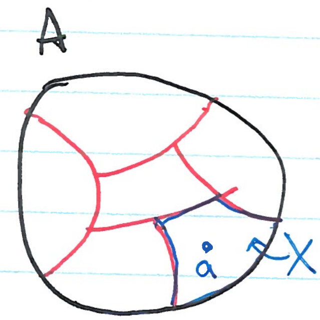
So $\{x : xRa\} \subseteq \{x : xRb\}$

The proof of the other direction: by symmetry

$aRb \Rightarrow bRa$

and now use a similar argument to show

$\{x : xRb\} \subseteq \{x : xRa\}$. ∎



Def: Suppose A is a set.

A partition $\mathcal{P}$ is a collection of (distinct) nonempty subsets of A, such that

- For every $a \in A$, $\exists X \in \mathcal{P}$ s.t. $a \in X$.
- For any two $X, Y \in \mathcal{P}$, (either $X = Y$, or $X \cap Y = \emptyset$)

Fact: If R is an equivalence relation on A, then the equivalence classes form a partition of A.

Fact: If $\mathcal{P}$ is any partition of A, then there's an equivalence relation on A whose equivalence classes are the elements of $\mathcal{P}$.

Exercise: Let $A = \mathbb{R}^2$. Consider the relation

$$(x_1, y_1) R (x_2, y_2) \text{ iff } \left(\begin{array}{l} x_1 - x_2 \in \mathbb{Z} \text{ and} \\ y_1 - y_2 \in \mathbb{Z}. \end{array} \right)$$

Come up with a description of the equivalence classes.

Answer: There is one equivalence class for every point $(a,b) \in [0,1) \times [0,1)$.